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Published in:
Journal of Functional Analysis

DOI:
[10.1016/j.jfa.2019.06.014](https://doi.org/10.1016/j.jfa.2019.06.014)

Publication date:
2019

Document version
Peer reviewed version

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Citation for published version (APA):
Brix, K. A., & Scarparo, E. (2019). C*-simplicity and representations of topological full groups of groupoids. *Journal of Functional Analysis*, 277(9), 2981-2996. <https://doi.org/10.1016/j.jfa.2019.06.014>

C*-SIMPLICITY AND REPRESENTATIONS OF TOPOLOGICAL FULL GROUPS OF GROUPOIDS

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ABSTRACT. Given an ample groupoid G with compact unit space, we study the canonical representation of the topological full group $[[G]]$ in the full groupoid C^* -algebra $C^*(G)$. In particular, we show that the image of this representation generates $C^*(G)$ if and only if $C^*(G)$ admits no tracial state. The techniques that we use include the notion of groups covering groupoids.

As an application, we provide sufficient conditions for C^* -simplicity of certain topological full groups, including those associated with topologically free and minimal actions of non-amenable and countable groups on the Cantor set.

1. INTRODUCTION

Topological full groups associated to group actions on the Cantor set have given rise to examples of groups with interesting new properties. See, e.g., [9] and [18] for recent developments. In the context of groupoids, the topological full group was introduced by H. Matui in [14], who investigated their relation with homology groups of groupoids.

Following a slightly different approach, V. Nekrashevych ([19]) defined the topological full group $[[G]]$ of an ample groupoid G with compact unit space to consist of the clopen bisections $U \subset G$ such that $r(U) = s(U) = G^{(0)}$. In this paper, we study the unitary representation $\pi: [[G]] \rightarrow C^*(G)$ given by $\pi(U) := 1_U$, for every $U \in [[G]]$. Let $C_\pi^*([[G]])$ denote the C^* -algebra generated by $\pi([[G]])$ in $C^*(G)$.

Our main result is as follows:

Theorem (Theorems 4.3 and 4.6). *Let G be an ample groupoid with compact unit space such that the orbit of each $x \in G^{(0)}$ has at least three points. Then $\overline{\text{span}}\{1 - 1_U \in C^*(G) \mid U \in [[G]]\}$ is a hereditary C^* -subalgebra of $C^*(G)$. Moreover, $C^*(G)$ admits no tracial state if and only if $C_\pi^*([[G]]) = C^*(G)$.*

This generalizes part of [8, Proposition 5.3] (see Remark 4.7). If, in addition, G is second countable, essentially principal and minimal, then $C_r^*(G)$ is stably isomorphic to $\overline{\text{span}}\{1 - 1_U \in C_r^*(G) \mid U \in [[G]]\}$ (Corollary 4.4).

Given an ample groupoid G with compact unit space, let π_r denote the canonical representation of $[[G]]$ in $C_r^*(G)$.

Recall that a group is said to be C^* -simple if its reduced C^* -algebra is simple. Recently, there has been a lot of progress in understanding this notion, and new characterizations of C^* -simplicity have been obtained (see [2], [10], [11]). In [12], A. Le Boudec and N. Matte Bon showed that a countable group of homeomorphisms

The first named author is supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92). The second named author is supported by CNPq, Brazil, 167983/2017-2.

on a Hausdorff space X is C^* -simple if the rigid stabilizers of non-empty and open subsets of X are non-amenable. By using this result, we show the following:

Theorem (Theorem 5.2). *Let G be a second countable, essentially principal, minimal and ample groupoid with compact unit space. If*

- (i) G is not amenable, or
- (ii) π_r does not weakly contain the trivial representation,

then $[[G]]$ is C^ -simple.*

Consequently, the topological full group associated with a topologically free and minimal action of a countable and non-amenable group on the Cantor set is C^* -simple (Corollary 5.4). For free actions, this was shown to be true in [12].

The paper is organized as follows. In Section 2, we collect basic definitions about groupoids, establish notation and present some relevant examples.

In Section 3, we study groups covering groupoids. Given an ample groupoid G with compact unit space, a subgroup $\Gamma \leq [[G]]$ is said to cover G if $G = \bigcup_{U \in \Gamma} U$. We investigate under which conditions $[[G]]$ covers G and show that $C_\pi^*([G])$ admits a character if and only if $G^{(0)}$ admits a G -invariant probability measure (Corollary 3.7).

In Section 4, we analyze the representation of the topological full group in the full and the reduced groupoid C^* -algebras to reach the main theorem above.

In Section 5, we apply the results of Sections 3 and 4 in order to study C^* -simplicity of the topological full group.

2. PRELIMINARIES

In this section we introduce relevant concepts and establish notation. Throughout the paper, we let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ denote the non-negative integers.

2.1. Ample groupoids. A topological groupoid G is *ample* if G is locally compact, Hausdorff, étale (in the sense that the range and source maps $r, s: G \rightarrow G$ are local homeomorphisms onto $G^{(0)}$) and the unit space $G^{(0)}$ is totally disconnected. The *orbit* of a point $x \in G^{(0)}$ is the set $G(x) := r(s^{-1}(x))$, and G is said to be *minimal* if $\overline{G(x)} = G^{(0)}$, for every $x \in G^{(0)}$.

A *bisection* is a subset $S \subset G$ such that $r|_S$ and $s|_S$ are injective. Note that, if S is open, then $r|_S$ and $s|_S$ are homeomorphisms onto their images. We will denote by \mathcal{S} the inverse semigroup of open bisections of G , and by $\mathcal{C} \subset \mathcal{S}$ the sub-inverse semigroup of compact open bisections. There is a homomorphism θ from \mathcal{S} to the inverse semigroup of homeomorphisms between open subsets of $G^{(0)}$, given by $\theta_U := r \circ (s|_U)^{-1}: s(U) \rightarrow r(U)$. As observed in [22], θ is injective if and only if G is *essentially principal* (that is, $\text{Int}\{g \in G : r(g) = s(g)\} = G^{(0)}$).

In the following we let $C_c(G)$ be the collection of complex valued, continuous and compactly supported functions on G . This is a $*$ -algebra with the convolution product

$$f \star g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta),$$

for $f, g \in C_c(G)$ and $\gamma \in G$, and $*$ -involution $f^*(\gamma) = \overline{f(\gamma^{-1})}$, for $f \in C_c(G)$ and $\gamma \in G$.

Let $C_r^*(G)$ and $C^*(G)$ denote the reduced and full groupoid C^* -algebras, respectively. For an introduction to (étale) groupoids and their C^* -algebras, the reader is referred to, e.g., [20] or [25].

If G is minimal and essentially principal, then $C_r^*(G)$ is simple (see, e.g., [25, Proposition 4.3.7]).

A regular Borel measure μ on $G^{(0)}$ is G -invariant if $\mu(r(S)) = \mu(s(S))$, for each $S \in \mathcal{S}$. Clearly, μ is G -invariant if and only if $\mu(r(U)) = \mu(s(U))$, for each $U \in \mathcal{C}$. The following proposition is well-known.

Proposition 2.1. *Let G be an ample groupoid with compact unit space. The following conditions are equivalent:*

- (i) $G^{(0)}$ admits a G -invariant probability measure;
- (ii) $C_r^*(G)$ admits a tracial state;
- (iii) $C^*(G)$ admits a tracial state.

Proof. The proof of the implications (i) \implies (ii) \implies (iii) can be found in [20, Theorem 3.4.4].

(iii) \implies (i): Let τ be a tracial state on $C^*(G)$. Given $U \in \mathcal{C}$, we have

$$\tau(1_{r(U)}) = \tau(1_U 1_{U^{-1}}) = \tau(1_{U^{-1}} 1_U) = \tau(1_{s(U)}).$$

Thus, the probability measure on $G^{(0)}$ induced by $\tau|_{C(G^{(0)})}$ is G -invariant. \square

Suppose $G^{(0)}$ admits a G -invariant measure μ . Then there is a representation $\rho: C_c(G) \rightarrow B(L^2(G^{(0)}, \mu))$ given by

$$(\rho(f)(\xi))(x) := \sum_{g \in r^{-1}(x)} f(g) \xi(s(g)), \quad (1)$$

for $f \in C_c(G)$, $\xi \in L^2(G^{(0)}, \mu)$ and $x \in G^{(0)}$.

Note that $\rho|_{C(G^{(0)})}$ is the representation by multiplication operators. Moreover, if U is a compact open bisection, then

$$(\rho(1_U)(\xi))(x) = \begin{cases} \xi(\theta_U^{-1}(x)), & x \in s(U), \\ 0, & x \notin s(U), \end{cases}$$

for $\xi \in L^2(G^{(0)}, \mu)$ and $x \in G^{(0)}$.

2.2. Topological full groups. Given an ample groupoid G with compact unit space, the *topological full group* of G is

$$[[G]] := \{U \in \mathcal{C} \mid r(U) = s(U) = G^{(0)}\}.$$

This definition coincides with the one from [19]. In [14], however, H. Matui defines the topological full group of G as $\theta([[G]])$. Therefore, if G is essentially principal then θ is injective and the two definitions coincide.

Two examples to have in mind are as follows.

Example 2.2. Let φ be an action of a group Γ on a compact Hausdorff space X . As a space, the *transformation groupoid* associated with φ is $G_\varphi := \Gamma \times X$ equipped with the product topology. The product of two elements $(h, y), (g, x) \in G_\varphi$ is defined if and only if $y = gx$ in which case $(h, gx)(g, x) := (hg, x)$. Inversion is given by $(g, x)^{-1} := (g^{-1}, gx)$. The unit space $G^{(0)}$ is naturally identified with X and G_φ is ample if X is totally disconnected.

The topological full group $[[G_\varphi]]$ consists of sets of the form $\bigcup_{i=1}^n \{g_i\} \times A_i$, where $g_1, \dots, g_n \in \Gamma$ and $A_1, \dots, A_n \subset X$ are clopen sets such that

$$X = \bigsqcup_{i=1}^n A_i = \bigsqcup_{i=1}^n g_i A_i.$$

In particular, there is a canonical injective homomorphism $\Gamma \longrightarrow [[G_\varphi]]$ sending $g \longmapsto \{g\} \times X$.

Example 2.3. Let $X := \{0, 1\}^{\mathbb{N}}$ be the full one-sided 2-shift and consider the Deaconu-Renault groupoid

$$G_{[2]} := \{(y, n, x) \in X \times \mathbb{Z} \times X \mid \exists l, k \in \mathbb{N} : n = l - k, y_{l+i} = x_{k+i} \forall i \in \mathbb{N}\}.$$

The product of $(z, n, y'), (y, m, x) \in G_{[2]}$ is well-defined if and only if $y' = y$ in which case $(z, n, y)(y, m, x) := (z, n + m, x)$. Inversion is given by $(y, n, x)^{-1} := (x, -n, y)$.

Let X_f be the set of finite words (including the empty word) on the alphabet $\{0, 1\}$. Given $\alpha \in X_f$, let $|\alpha|$ denote its length and let $\overline{\alpha} := \{x \in X \mid x_i = \alpha_i, 0 \leq i < |\alpha|\}$ be the cylinder set of α . The topology on $G_{[2]}$ is generated by sets of the form

$$Z(\beta, \alpha) := \{(y, |\beta| - |\alpha|, x) \in G_{[2]} \mid y \in \overline{\beta}, x \in \overline{\alpha}, y_{|\beta|+i} = x_{|\alpha|+i} \forall i \in \mathbb{N}\},$$

for $\alpha, \beta \in X_f$. This topology is strictly finer than the one inherited from the product topology and $G_{[2]}$ is ample with compact unit space. Note as well that $G_{[2]}$ is minimal.

The topological full group $[[G_{[2]}]]$ consists of sets of the form

$$\bigcup_{j=1}^n Z(\beta^j, \alpha^j), \tag{2}$$

with $X = \bigsqcup_{j=1}^n \overline{\alpha^j} = \bigsqcup_{j=1}^n \overline{\beta^j}$.

We would now like to recall the isomorphism between Thompson's group V and $[[G_{[2]}]]$, observed in [15] (see also [16] and [17]).

Thompson's group V consists of piecewise linear, right continuous bijections on $[0, 1)$ which have finitely many points of non-differentiability, all being dyadic rationals, and have a derivative which is a power of 2 at each point of differentiability.

Given $\alpha, \beta \in X_f$, let $\psi(\alpha) := \sum_i \alpha_i 2^{-i} \in [0, 1)$ and $I(\alpha) := [\psi(\alpha), \psi(\alpha) + 2^{-|\alpha|}]$. The isomorphism from $[[G_{[2]}]]$ to V takes $\bigcup_j Z(\beta^j, \alpha^j)$ as in (2) and sends it to the bijection on $[0, 1)$ which, restricted to $I(\alpha^j)$, is linear, increasing and onto $I(\beta^j)$, for every j .

The next example shows that the short exact sequence induced by the quotient $\theta: [[G]] \longrightarrow \theta([[G]])$ is not always split. Since we are interested in studying the canonical representation of $[[G]]$ in $C^*(G)$, this illustrates why we have chosen to treat the topological full group as bisections, rather than homeomorphisms on the unit space.

Example 2.4. Let $X := \mathbb{Z} \cup \{\infty\}$ be the one-point compactification of \mathbb{Z} and define an action $\varphi: \mathbb{Z} \curvearrowright X$ by

$$\varphi_n(x) := \begin{cases} (-1)^n x, & x \in \mathbb{Z}, \\ \infty, & x = \infty, \end{cases}$$

for $n \in \mathbb{Z}$. Note that $\{1\} \times X$ is a compact open bisection in the transformation groupoid G_φ and that the homeomorphism

$$\theta_{\{1\} \times X}(x) = \begin{cases} -x, & x \in \mathbb{Z}, \\ \infty, & x = \infty, \end{cases}$$

for $x \in X$, has order 2.

Moreover, for any $U \in [[G_\varphi]]$ satisfying $\theta_U = \theta_{\{1\} \times X}$, there is an odd integer n such that $(n, \infty) \in U$. In particular, U has infinite order. Therefore, the short exact sequence induced by $\theta: [[G_\varphi]] \rightarrow \theta([[G_\varphi]])$ is not split.

2.3. Unitary representations. Let G be an ample groupoid with compact unit space. There is a unitary representation

$$\begin{aligned} \pi: [[G]] &\rightarrow C^*(G) \\ U &\mapsto 1_U, \end{aligned}$$

We will denote the analogous representation of $[[G]]$ in $C_r^*(G)$ by π_r .

If σ and η are unitary representations of a group Γ on unital C^* -algebras, then σ is said to *weakly contain* η if

$$\left\| \sum_i \alpha_i \eta(g_i) \right\| \leq \left\| \sum_i \alpha_i \sigma(g_i) \right\|,$$

for every $\sum_i \alpha_i g_i \in \mathbb{C}\Gamma$. The *trivial representation* $\Gamma \rightarrow \mathbb{C}$ satisfies $g \mapsto 1$, for every $g \in \Gamma$.

Given a unitary representation η of Γ on a unital C^* -algebra A , we denote by $C_\eta^*(\Gamma)$ the C^* -algebra generated by the image of η . Note that if η weakly contains the trivial representation, then $C_\eta^*(\Gamma)$ admits a character whose kernel is $\overline{\text{span}}\{1_A - \eta(g) \mid g \in \Gamma\}$.

Proposition 2.5. *Let η be a unitary representation of a group Γ on a unital C^* -algebra A . Then η weakly contains the trivial representation if and only if $1_A \notin \overline{\text{span}}\{1_A - \eta(g) \mid g \in \Gamma\}$.*

Proof. The forward implication is evident, so we only prove the backward one.

Let $B := \overline{\text{span}}\{1_A - \eta(g) : g \in \Gamma\}$. If $1_A \notin B$, then, since B is a C^* -algebra, $\text{dist}(1_A, B) = 1$. Hence, for every $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and $g_1, \dots, g_n \in \Gamma$, we have that

$$\left\| \sum \alpha_i \eta(g_i) \right\| = \left\| \left(\sum \alpha_i \right) \cdot 1_A - \sum \alpha_i (1_A - \eta(g_i)) \right\| \geq \left| \sum \alpha_i \right|,$$

thus showing that η weakly contains the trivial representation. \square

3. GROUPS COVERING GROUPOIDS

An ample groupoid G can always be covered by compact open bisections. We investigate to which degree G can be covered by compact open bisections U which satisfy $r(U) = s(U) = G^{(0)}$. We show that if $\Gamma \leq [[G]]$ covers G and μ is a Γ -invariant probability measure on $G^{(0)}$, then μ is also G -invariant.

Definition 3.1. Given an ample groupoid G with compact unit space, we say that a subgroup $\Gamma \leq [[G]]$ *covers* G if $G = \bigcup_{U \in \Gamma} U$.

The idea of covering a groupoid G by compact open bisections U such that $r(U) = s(U) = G^{(0)}$ has already appeared in H. Matui's study of automorphisms of G , cf. [14, Proposition 5.7].

If G is essentially principal, then a subgroup $\Gamma \leq [[G]]$ covers G if and only if, for each open bisection S and $x \in s(S)$, there are $U \in \Gamma$ and a neighborhood $W \subset s(S)$ of x such that $\theta_U|_W = \theta_S|_W$.

Example 3.2. If φ is an action of a group Γ on a compact Hausdorff and totally disconnected space, then the copy of Γ in $[[G_\varphi]]$ covers G_φ .

Example 3.3. Recall that Thompson's group $T < V$ consists of the elements of Thompson's group V (see Example 2.3) which have at most one point of discontinuity.

Let $G_{[2]}$ be the groupoid of Example 2.3. Under the identification of V with $[[G_{[2]}]]$, T covers $G_{[2]}$. This follows from the fact that if $I, J \subset [0, 1)$ are left-closed and right-open intervals with endpoints in $\mathbb{Z}[1/2]$, then there exists a piecewise linear homeomorphism $f: I \rightarrow J$ with a derivative which is a power of 2 at each point of differentiability and with finitely many points of non-differentiability, all of which belong to $\mathbb{Z}[1/2]$.

Lemma 3.4. *Let G be an ample groupoid with compact unit space. If $|G(x)| \geq 2$ for every $x \in G^{(0)}$, then $[[G]]$ covers G .*

Proof. Let $g \in G$. If $r(g) \neq s(g)$, then there is a compact open bisection V containing g and such that $s(V) \cap r(V) = \emptyset$. Let $U := V \cup V^{-1} \cup (G^{(0)} \setminus (s(V) \cup r(V)))$. Then $g \in U \in [[G]]$.

If $r(g) = s(g)$, then there is $h \in s^{-1}(r(g))$ such that $r(hg) = r(h) \neq s(h) = s(hg)$ since $|G(r(g))| \geq 2$. As before, there are $U, U' \in [[G]]$ such that $h \in U$ and $hg \in U'$. Hence, $g \in U^{-1}U' \in [[G]]$. \square

The purpose of the next example is to show that the above result may fail if one does not make any assumption on the orbits.

Example 3.5. Consider $X := \mathbb{Z} \cup \{\pm\infty\}$ equipped with the order topology and let $\varphi: \mathbb{Z} \curvearrowright X$ be the action given by $\varphi_t(x) := t + x$, for $t \in \mathbb{Z}$ and $x \in X$. The transformation groupoid G_φ is ample with compact unit space.

Given $x, z \in X$ we put $[x, z] := \{y \in X : x \leq y \leq z\}$. Then

$$H := \{(t, x) \in \mathbb{Z} \times [0, +\infty) : -t \leq x\}$$

is an ample subgroupoid of G_φ . Incidentally, this is the groupoid of the partial action obtained by restricting φ to $[0, +\infty)$ (see [7] and [13] for more details). Observe that $|H((0, +\infty))| = 1$.

We claim that if $U \in [[H]]$, then $(1, +\infty) \notin U$. Otherwise, there is $t \in \mathbb{N}$ such that $S := \{1\} \times [t, +\infty) \subset U$ and $U \setminus S \in \mathcal{C}$. But then $s(U \setminus S) = [0, t-1]$ and $r(U \setminus S) = [0, t]$ contradicting the fact that r and s are injective on $U \setminus S$. Hence, $(1, +\infty) \notin U$ and $[[H]]$ does not cover H .

Recall that a probability measure μ on $G^{(0)}$ is G -invariant if $\mu(s(S)) = \mu(r(S))$ for every $S \in \mathcal{S}$. Moreover, if $\Gamma \leq [[G]]$, then we say μ is Γ -invariant if it is invariant with respect to the action θ .

Proposition 3.6. *Let G be an ample groupoid with compact unit space and Γ a subgroup of $[[G]]$. Consider the following conditions:*

- (i) $G^{(0)}$ admits a G -invariant probability measure;
- (ii) $\pi|_\Gamma$ weakly contains the trivial representation;
- (iii) $C_\pi^*(\Gamma)$ admits a character;
- (iv) $G^{(0)}$ admits a Γ -invariant probability measure.

Then (i) \implies (ii) \implies (iii) \implies (iv). If Γ covers G , then (iv) \implies (i) and all conditions are equivalent.

Proof. (i) \implies (ii): Suppose μ is a G -invariant measure on $G^{(0)}$ and let $\rho: C_c(G) \rightarrow B(L^2(G^{(0)}, \mu))$ be the representation given by (1). The vector $1_{G^{(0)}} \in L^2(G^{(0)}, \mu)$ is invariant for the representation $\rho \circ \pi|_\Gamma: \Gamma \rightarrow B(L^2(G^{(0)}, \mu))$. Hence, $\pi|_\Gamma$ weakly contains the trivial representation.

The implication (ii) \implies (iii) is evident.

(iii) \implies (iv): Let φ be a character on $C_\pi^*(\Gamma)$ and τ a state on $C^*(G)$ which is an extension of φ . Then $C_\pi^*(\Gamma)$ is in the multiplicative domain of τ . Clearly, $\tau|_{C(G^{(0)})}$ induces a Γ -invariant probability measure on $G^{(0)}$.

Now, suppose Γ covers $[[G]]$ and let us show that (iv) \implies (i). Let μ be a Γ -invariant probability measure on $G^{(0)}$. We claim that μ is also G -invariant. Indeed, since Γ covers G , given $S \in \mathcal{C}$, we have that $S = \bigcup_{U \in \Gamma} (S \cap U)$. As S is compact, there are $S_1, \dots, S_n \in \mathcal{C}$ and $U_1, \dots, U_n \in \Gamma$ such that $S = \bigsqcup_i S_i$ and $S_i \subset U_i$ for $1 \leq i \leq n$. In particular, $\theta_{U_i}(s(S_i)) = r(S_i)$ for every i . It follows that

$$\mu(r(S)) = \sum_{i=1}^n \mu(r(S_i)) = \sum_{i=1}^n \mu(s(S_i)) = \mu(s(S)).$$

Therefore, μ is a G -invariant probability measure on $G^{(0)}$. \square

Corollary 3.7. *Let G be an ample groupoid with compact unit space. The following conditions are equivalent:*

- (i) $G^{(0)}$ admits a G -invariant probability measure;
- (ii) π weakly contains the trivial representation;
- (iii) $C_\pi^*([G])$ admits a character;
- (iv) $G^{(0)}$ admits a $[[G]]$ -invariant probability measure.

Proof. The implications (i) \implies (ii) \implies (iii) \implies (iv) follow from Proposition 3.6.

(iv) \implies (i): If, for each $x \in G^{(0)}$, $|G(x)| \geq 2$, then the result follows from Lemma 3.4 and Proposition 3.6.

If there is $x \in X$ such that $|G(x)| = 1$, then point evaluation at x is a G -invariant probability measure. \square

4. REPRESENTATIONS OF TOPOLOGICAL FULL GROUPS

In this section, we prove the main results of the article. We start with two technical lemmas.

Lemma 4.1. *Let G be an ample groupoid with compact unit space. If $S, T \in [[G]]$ and $W \subset G^{(0)}$ is a clopen subset such that $\theta_S(W), W, \theta_T^{-1}(W)$ are mutually disjoint, then $(1 - 1_S)1_W(1 - 1_T) \in \text{span}\{1 - 1_U \in C_c(G) \mid U \in [[G]]\}$.*

Proof. We have

$$\begin{aligned} (1 - 1_S)1_W(1 - 1_T) &= 1_{SWT} + 1_{T^{-1}WS^{-1}} + 1_W + 1_{G^{(0)} \setminus (\theta_S(W) \cup W \cup \theta_T^{-1}(W))} \\ &\quad - (1_{T^{-1}WS^{-1}} + 1_{G^{(0)} \setminus (\theta_S(W) \cup W \cup \theta_T^{-1}(W))} + 1_{SW} + 1_{WT}). \end{aligned}$$

The sets $SWT, T^{-1}WS^{-1}, W$ and $G^{(0)} \setminus (\theta_T^{-1}(W) \cup W \cup \theta_S(W))$ are mutually disjoint and their union is in $[[G]]$. This is also the case for the sets $T^{-1}WS^{-1}, SW, WT$ and $G^{(0)} \setminus (\theta_S(W) \cup W \cup \theta_T^{-1}(W))$ and so the result follows. \square

In order to employ Lemma 4.1, the following result will be useful.

Lemma 4.2. *Let G be an ample groupoid with compact unit space. If $x \in G^{(0)}$ and $y \in G(x) \setminus \{x\}$, then*

$$\text{span}\{1 - 1_U \mid U \in [[G]]\} = \text{span}\{1_L(1 - 1_S) \mid S, L \in [[G]], \theta_S(x) = y\} \quad (3)$$

$$= \text{span}\{(1 - 1_T)1_R \mid T, R \in [[G]], \theta_T^{-1}(x) = y\}. \quad (4)$$

Proof. Let

$$B := \text{span}\{1_L(1 - 1_S) \mid S, L \in [[G]], \theta_S(x) = y\}$$

and take $U \in [[G]]$. We will show that $1 - 1_U \in B$.

If $\theta_U(x) = x$, we take $L \in [[G]]$ such that $\theta_{LU}(x) = \theta_L(x) = y$. Then $1 - 1_U = 1_{L^{-1}}(1_L - 1) + 1_{L^{-1}}(1 - 1_{LU}) \in B$.

On the other hand, if $\theta_U(x) \neq x$, we take $L \in [[G]]$ such that $\theta_L(x) = x$ and $\theta_{LU}(x) = y$. Then $\theta_{L^{-1}}(x) = x$ so $1 - 1_{L^{-1}} \in B$ by the above. Hence $1 - 1_U = (1 - 1_{L^{-1}}) + 1_{L^{-1}}(1 - 1_{LU}) \in B$ proving (3).

By taking adjoints and interchanging x and y , the equality in (4) follows from (3). \square

The next result generalizes [24, Theorem 3.7], which was obtained in the setting of Cantor minimal \mathbb{Z} -systems.

Theorem 4.3. *Let G be an ample groupoid with compact unit space. If $|G(x)| \geq 3$ for every $x \in G^{(0)}$, then $\overline{\text{span}}\{1 - 1_U \in C^*(G) \mid U \in [[G]]\}$ is a hereditary C^* -subalgebra of $C^*(G)$.*

Proof. Let $B := \overline{\text{span}}\{1 - 1_U \in C^*(G) \mid U \in [[G]]\}$. We will first show that

$$BC(G^{(0)})B \subset B. \quad (5)$$

It suffices to prove that, given $U, V \in [[G]]$, there is a basis \mathcal{W} for $G^{(0)}$ consisting of compact open sets satisfying $(1 - 1_U)1_W(1 - 1_V) \in B$, for each $W \in \mathcal{W}$. Take $x \in G^{(0)}$ and let y and z be distinct elements in $G(x) \setminus \{x\}$. By Lemma 4.2, there are $n \in \mathbb{N}$ and $L_1, \dots, L_n, U_1, \dots, U_n, V_1, \dots, V_n$ and R_1, \dots, R_n in $[[G]]$ and $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{C}$ such that

$$1 - 1_U = \sum_{i=1}^n \alpha_i 1_{L_i}(1 - U_i), \quad 1 - 1_V = \sum_{i=1}^n \beta_i (1 - 1_{V_i}) R_i$$

with $\theta_{U_i}(x) = y$ and $\theta_{V_i}^{-1}(x) = z$ for every $i = 1, \dots, n$. By Lemma 4.1, we see that $(1 - 1_U)1_W(1 - 1_V) \in B$ for every sufficiently small compact open neighborhood W of x . This proves (5).

Next we show that $BC^*(G)B \subset B$. It suffices to prove that $B1_W B \subset B$, for every W in a basis for G consisting of compact open sets. Given $g \in G$, take $U \in [[G]]$ such that $\theta_U(r(g)) \neq s(g)$. Then, for $W \subset G^{(0)}$ sufficiently small compact open neighborhood of g , we have that $\theta_U(r(W)) \cap s(W) = \emptyset$. Let

$$V := UW \cup (UW)^{-1} \cup (G^{(0)} \setminus (\theta_U(r(W)) \cup s(W))) \in [[G]].$$

Since $\theta_U(r(W)) \cap s(W) = \emptyset$, we have $UWV = \theta_U(r(W)) \subset G^{(0)}$ and, finally,

$$B1_W B = B(1_U 1_W 1_V)B = B1_{\theta_U(r(W))} B \subset B$$

by (5). \square

Corollary 4.4. *Let G be an ample groupoid with compact unit space. If $|G(x)| \geq 3$ for every $x \in G^{(0)}$, then $\overline{\text{span}}\{1 - 1_U \in C_r^*(G) \mid U \in [[G]]\}$ is a hereditary C^* -subalgebra of $C_r^*(G)$. If, in addition, G is second countable, essentially principal and minimal, then $\overline{\text{span}}\{1 - 1_U \in C_r^*(G) \mid U \in [[G]]\}$ is stably isomorphic to $C_r^*(G)$.*

Proof. The first assertion follows directly from Theorem 4.3 while the second follows from simplicity of $C_r^*(G)$ and Brown's theorem [3, Theorem 2.8]. \square

The next example shows that Theorem 4.3 does not hold without the hypothesis on orbits.

Example 4.5. Let $X := \mathbb{Z} \cup \{\pm\infty\}$ with the order topology and let φ be the action of the infinite dihedral group $\mathbb{Z} \rtimes \mathbb{Z}_2$ on X given by $\varphi_{(n,j)}(x) := n + (-1)^j x$, for $(n,j) \in \mathbb{Z} \rtimes \mathbb{Z}_2$ and $x \in X$. Then $|G_\varphi(x)| \geq 2$ for every $x \in G_\varphi^{(0)}$.

By arguing as in Example 3.5, one concludes that, given $U \in [[G_\varphi]]$, there is $(n,j) \in \mathbb{Z} \rtimes \mathbb{Z}_2$ such that $((n,j), \pm\infty) \in U$.

Let $E: C^*(G_\varphi) \rightarrow C(G_\varphi^{(0)})$ be the canonical conditional expectation and let $\delta_{+\infty}$ and $\delta_{-\infty}$ be the two states on $C(G_\varphi^{(0)})$ given by point-evaluations at $+\infty$ and $-\infty$, respectively. Then $\delta_{+\infty} \circ E$ and $\delta_{-\infty} \circ E$ are two distinct states on $C^*(G_\varphi)$ whose restrictions to $B = \overline{\text{span}}\{1 - 1_U \in C^*(G_\varphi) \mid U \in [[G_\varphi]]\}$ agree. Hence, B is not a hereditary C^* -subalgebra of $C^*(G_\varphi)$.

By combining Theorem 4.3 with the results of the previous section, we obtain the following:

Theorem 4.6. *Let G be an ample groupoid with compact unit space. Assume that $|G(x)| \geq 3$ for every $x \in G^{(0)}$. The following conditions are equivalent.*

- (i) $C^*(G)$ admits no tracial state;
- (ii) $C_\pi^*([[G]])$ admits no character;
- (iii) π does not weakly contain the trivial representation;
- (iv) $C_\pi^*([[G]]) = C^*(G)$.

Proof. The equivalences (i) \iff (ii) \iff (iii) follow from Proposition 2.1 and Corollary 3.7.

(iii) \implies (iv): By Proposition 2.5 and Theorem 4.3, $B := \overline{\text{span}}\{1 - 1_U \mid U \in [[G]]\}$ is a hereditary C^* -subalgebra of $C^*(G)$ and $1_{C^*(G)} \in B$. Hence, $B = C^*(G)$. Since $B \subset C_\pi^*([[G]])$ the result follows.

(iv) \implies (i): If $C^*(G)$ has a tracial state, then $G^{(0)}$ admits an invariant probability measure μ , cf. Proposition 2.1. Since $|G(x)| > 1$ for each $x \in G^{(0)}$, μ cannot be a point-evaluation. Let ρ be the representation of $C_c(G)$ in $B(L^2(G^{(0)}, \mu))$ as in (1). Then ρ extends to a representation of $C^*(G)$ and of $C_\pi^*([[G]])$. Note that the vector $1_{G^{(0)}} \in L^2(G^{(0)}, \mu)$ is invariant under $\rho(\pi([[G]]))$ and thus under $\rho|_{C_\pi^*([[G]])}$. Now, if $C_\pi^*([[G]]) = C^*(G)$, then $C(G^{(0)}) \subset C_\pi^*([[G]])$ but $\mathbb{C}1_{G^{(0)}}$ is not invariant under $\rho|_{C(G^{(0)})}$. Indeed, if $X \subset G^{(0)}$ is any proper, non-empty subset which is compact and open, then $\rho(1_X)(1_{G^{(0)}}) = 1_X$. Therefore $C_\pi^*([[G]]) \neq C^*(G)$. \square

Remark 4.7. In [8, Proposition 5.3], U. Haagerup and K. Olesen considered a certain representation σ of Thompson's group V in the Cuntz algebra \mathcal{O}_2 and showed that $C_\sigma^*(V) = \mathcal{O}_2$. Under the identifications of V with $[[G_{[2]}]]$ (see Example 2.3) and \mathcal{O}_2 with $C^*(G_{[2]})$, one can check that σ and π coincide. Hence, Theorem 4.6 recovers part of U. Haagerup and K. Olesen's result.

We now state and prove a version of Theorem 4.6 regarding $C_r^*(G)$.

Theorem 4.8. *Let G be an ample groupoid with compact unit space. Assume that $|G(x)| \geq 3$ for each $x \in G^{(0)}$ and consider the following conditions:*

- (i) $C_r^*(G)$ admits no tracial state;
- (ii) $C_{\pi_r}^*([G])$ admits no character;
- (iii) π_r does not weakly contain the trivial representation;
- (iv) $C_{\pi_r}^*([G]) = C_r^*(G)$.

Then (i) \implies (ii) \iff (iii) \iff (iv).

Proof. The implications (i) \implies (ii) \implies (iii) \implies (iv) are done as in the full case.

(iv) \implies (ii). If $C_{\pi_r}^*(G) = C_r^*(G)$ admits a character τ , then $\tau|_{C(G^{(0)})}$ is a point evaluation at some $x \in G^{(0)}$. As τ is a tracial state, it follows that for each compact and open bisection S with $x \in s(S)$, we have $\theta_S(x) = x$. This contradicts the hypothesis that $|G(x)| > 1$. \square

The next example shows that the implication from (ii) to (i) in the above theorem fails in general, even in the case when G is a principal, minimal and ample groupoid with unit space homeomorphic to the Cantor set.

Example 4.9. Let Γ be a non-amenable, countable and residually finite group. There is a descending sequence $(\Gamma_n)_n$ of finite-index normal subgroups of Γ such that the canonical map $j: \Gamma \longrightarrow \prod \frac{\Gamma}{\Gamma_n}$ is injective. Then $X := \overline{j(\Gamma)}$ is a topological group homeomorphic to the Cantor set. Furthermore, the action φ by multiplication of Γ on X is free, minimal and the Haar measure on X is Γ -invariant (actions of this sort were studied in detail in [5]).

Then $C_r^*(G_\varphi)$ admits a tracial state, whereas $C_{\pi_r}^*([G_\varphi])$ does not admit a character, since $C_r^*(\Gamma)$ embeds unitally in it and Γ is non-amenable.

5. C^* -SIMPLICITY OF TOPOLOGICAL FULL GROUPS

As an application of the above results, we provide conditions which ensure that the topological full group of an ample groupoid is C^* -simple.

Recall that an ample groupoid G is *amenable* if there exists a net $(\mu_i)_i$ in $C_c(G)$ of non-negative functions such that

$$\sum_{h \in s^{-1}(r(g))} \mu_i(h) \longrightarrow 1 \quad \text{and} \quad \sum_{h \in s^{-1}(r(g))} |\mu_i(h) - \mu_i(hg)| \longrightarrow 0, \quad (6)$$

for $g \in G$, uniformly on compact subsets of G . Amenability of G is equivalent to nuclearity of $C_r^*(G)$, and it implies that $C^*(G)$ and $C_r^*(G)$ are canonically isomorphic. For a proof of these facts, see, e.g., [4] and [23]. R. Willett constructed in [26] an example of non-amenable groupoid G such that $C^*(G)$ is canonically isomorphic to $C_r^*(G)$.

Lemma 5.1. *Let G be an ample groupoid with compact unit space. If $\Gamma \leq [G]$ is an amenable subgroup which covers G , then G is amenable.*

Proof. We are going to construct functions satisfying (6). Let $K \subset G$ be a compact subset and let $\epsilon > 0$. As Γ covers G and is amenable, there are $V_1, \dots, V_n \in \Gamma$ such that $K \subset \bigcup_{i=1}^n V_i$ and a finite subset $F \subset \Gamma$ such that

$$\frac{|F \Delta FV_i|}{|F|} < \epsilon,$$

for $1 \leq i \leq n$. Let $\mu := \frac{1}{|F|} \sum_{U \in F} 1_U$. For $x \in G^{(0)}$ we have $\sum_{h \in s^{-1}(x)} \mu(h) = 1$. Given $g \in K$, take V_i such that $g \in V_i$. Then, for $h \in s^{-1}(r(g))$,

$$\begin{aligned} |F| |\mu(h) - \mu(hg)| &= \left| \sum_{U \in F} 1_U(h) - 1_{UV_i^{-1}}(h) \right| \\ &\leq \left| \sum_{U \in F \setminus (FV_i^{-1})} 1_U(h) - \sum_{U \in F \setminus (FV_i)} 1_{UV_i^{-1}}(h) \right| \\ &\quad + \left| \sum_{U \in F \cap FV_i^{-1}} 1_U(h) - \sum_{U \in F \cap FV_i} 1_{UV_i^{-1}}(h) \right| \\ &\leq \sum_{U \in F \setminus (FV_i^{-1})} 1_U(h) + \sum_{U \in F \setminus (FV_i)} 1_{UV_i^{-1}}(h). \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{h \in s^{-1}(r(g))} |\mu(h) - \mu(hg)| &\leq \frac{1}{|F|} \sum_h \left(\sum_{U \in F \setminus (FV_i^{-1})} 1_U(h) + \sum_{U \in F \setminus (FV_i)} 1_{UV_i^{-1}}(h) \right) \\ &= \frac{|F \setminus (FV_i^{-1})| + |F \setminus (FV_i)|}{|F|} < \epsilon. \end{aligned}$$

Therefore, G is amenable. \square

The converse implication is not true in general, see, e.g., Remark 5.5.

Suppose a group Γ is acting on a set X and let $U \subset X$ be a subset. The *rigid stabilizer* of U with respect to the action is the subgroup $\Gamma_U \leq \Gamma$ of the elements which pointwise fix the complement $X \setminus U$. Let $W \subset G^{(0)}$ be non-empty and clopen and let $G_W = r^{-1}(W) \cap s^{-1}(W)$ be the restricted groupoid. If $[[G]]_W$ is the rigid stabilizer of W with respect to the action $\theta: [[G]] \curvearrowright G^{(0)}$, then there is a surjective homomorphism $[[G]]_W \rightarrow [[G_W]]$ given by restriction. If G is essentially principal this map is an isomorphism.

Theorem 5.2. *Let G be a second countable, essentially principal, minimal and ample groupoid with compact unit space. If*

- (i) G is not amenable, or
- (ii) π_r does not weakly contain the trivial representation,

then $[[G]]$ is C^ -simple.*

Proof. Assume $[[G]]$ is not C^* -simple. By [12, Theorem 3.7], there exists a non-empty and clopen $W \subset G^{(0)}$ such that the rigid stabilizer $[[G]]_W \cong [[G_W]]$ is amenable. Clearly, G_W is an essentially principal, minimal and ample groupoid with compact unit space. By Lemma 5.1, G_W is thus amenable and $C_r^*(G_W)$ is nuclear.

Since $C_r^*(G)$ is simple, the projection $1_W \in C_r^*(G)$ is full. It therefore follows from [14, Lemma 5.2] and Brown's theorem ([3, Theorem 2.8]) that the full corner $C_r^*(G_W) = 1_W C_r^*(G) 1_W$ is stably isomorphic to $C_r^*(G)$. Consequently, $C_r^*(G)$ is nuclear, and G is amenable.

Furthermore, amenability of $[[G_W]]$ implies that W admits a $[[G_W]]$ -invariant probability measure. Corollary 3.7 and Proposition 2.1 then imply that $C_r^*(G_W)$ admits a tracial state. As $C_r^*(G_W)$ is simple, the tracial state is faithful. Hence, $C_r^*(G_W)$ is stably finite and, consequently, so is $C_r^*(G)$.

Now, [21, Theorem 6.5] (or [1, Theorem 5.14]) implies that $C_r^*(G)$ admits a tracial state. Since $C_r^*(G) = C^*(G)$, we conclude from Corollary 3.7 again that $\pi = \pi_r$ weakly contains the trivial representation. \square

The next corollary is an immediate consequence of Theorems 4.8 and 5.2.

Corollary 5.3. *Let G be a second countable, essentially principal, minimal and ample groupoid with compact unit space. If $C_r^*(G)$ admits no tracial state, then $[[G]]$ is C^* -simple.*

Recall that an action of a group Γ on a topological space X is *topologically free* if $\text{Int}\{x \in X \mid gx = x\} = \emptyset$, for each $g \in \Gamma \setminus \{e\}$. The following result generalizes [12, Theorem 4.38], which assumed freeness of the action.

Corollary 5.4. *Let φ be a topologically free and minimal action of a countable and non-amenable group Γ on the Cantor set. Then $[[G_\varphi]]$ is C^* -simple.*

Proof. Since $C_r^*(\Gamma)$ embeds unitally in $C_{\pi_r}^*([G_\varphi])$, non-amenability of Γ implies that π_r does not weakly contain the trivial representation. \square

Remark 5.5. In [6], G. Elek and N. Monod constructed a free and minimal action φ of \mathbb{Z}^2 on the Cantor set such that $[[G_\varphi]]$ is not amenable. This example is not covered by Theorem 5.2, and we do not know whether $[[G_\varphi]]$ is C^* -simple.

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